NPS-EC-94-004

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Efficient Parabolic Equation Solution of Radiowave Propagation in an Inhomogeneous Atmosphere and Over Irregular Terrain: Formulation

by

Ramakrishna Janaswamy

June 1994

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Prepared for: Naval Postgraduate School, Monterey, CA

Naval Postgraduate School
Monterey, California 93943-5000

Rear Admiral T. A. Mercer
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Department of Electrical and
Computer Engineering

REPORT DOCUMENTATION PAGE	Form Approved OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
June 1994

3. REPORT TYPE AND DATES COVERED
Technical Report 1 Jan 94–1 Jun 94

4. TITLE AND SUBTITLE
Efficient Parabolic Equation Solution of Radiowave Propagation in an Inhomogeneous Atmosphere and Over Irregular Terrain: Formulation

5. FUNDING NUMBERS

6. AUTHOR(S)
Ramakrishna Janaswamy

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Postgraduate School
Monterey, CA 93943-5000

8. PERFORMING ORGANIZATION REPORT NUMBER
NPS-EC-94-004

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Naval Postgraduate School
Monterey, CA 93943

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES
The views expressed in this report are those of the authors and do not reflect the official policy or position of the Department of Defense or the United States Government.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release;
distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
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14. SUBJECT TERMS
Radiowave Propagation, Parabolic Equation, Finite Differences

15. NUMBER OF PAGES
37

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

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1. Introduction

It is well known that multipath fading can significantly effect the link reliability in a communications system or target detectability in the case of radar. The path between a transmitted and a receiver is often obstructed by natrual or man-made obstacles such as hills, buildings, atmospheric layers, trees, rain, fog, etc. Propagation outage due to multipath fading depends in a complicated manner on propagation climate, terrain features, path length, radio frequency, and fade margin. In the case of atmospheric multipath fading, interference due to two or more super-refracted rays arriving at the receiver via different paths can lead to a complete loss of signal. Other pathological phenomenon such as obstruction fading (caused by sub-refractive atmospheric effects) and ducting (caused by extreme super-refractive effects) are also possible. Such phenomenon are more common in warm and tropical climates, particularly near shores, where elevated inversions are formed easily due to the large temperature and partial pressure differentials. Reflection multipath fading, which is due to interference between the direct and the ground reflected ray depends strongly on the terrain geometry and ground constants. Moreover, elevated terrain features could completely mask a receiver from a transmitter leading to severe loss of signal (in some cases, it is advantageous to site antennas behind hills to provide shielding against undesirable interference). It is very important to assess the effects of environment on the link. A computer model that can take into account a given refractive index profile, terrain elevation data, and varying ground parameters will be very helpful in predicting the link performance.

In this report we present formulation details for an efficient numerical solution of wave propagation in an inhomogeneous atmosphere and over irregular terrain using parabolic equation.

Unlike all other previous formulations of the parabolic equation, we will use a modified Helmholz equation for propagation in an inhomogeneous atmosphere as suggested by Maxwell's equations (all previous formulations use a Helmholtz equation which is only true for fields in a homogeneous medium). Because the parabolic equation is a full-wave method, it will include all aspects of wave propagation such as reflection, refraction, diffraction, and surface wave propagation. In this respect it is far superior to the commonly used ray method.

Parabolic equation approximation to an elliptic partial differential equation, which the true fields satisfy, has proven to be a viable approach for studying propagation problems in underwater acoustics. The method is just gaining popularity with the electromagnetic community. Although the parabolic equation regards waves as essentially traveling one-way, it allows a rapid solution of the fields by way of marching along the range starting from an initial range. Another advantage of the PE method compared to the ray methods is that it is valid even in the shadow region where the simple ray methods completely break down. Furthermore, it appears to be the only practical method for predicting propagation over long ranges (greater than 1 km) over a wideband from HF (a few MHz) through SHF (a few tens of GHz). The method is, however, not without limitations. In its standard form, the accuracy of the method is limited to waves traveling essentially within $\pm 10°$ from horizontal. Furthermore, treatment of the boundary conditions on the uneven terrain is difficult.

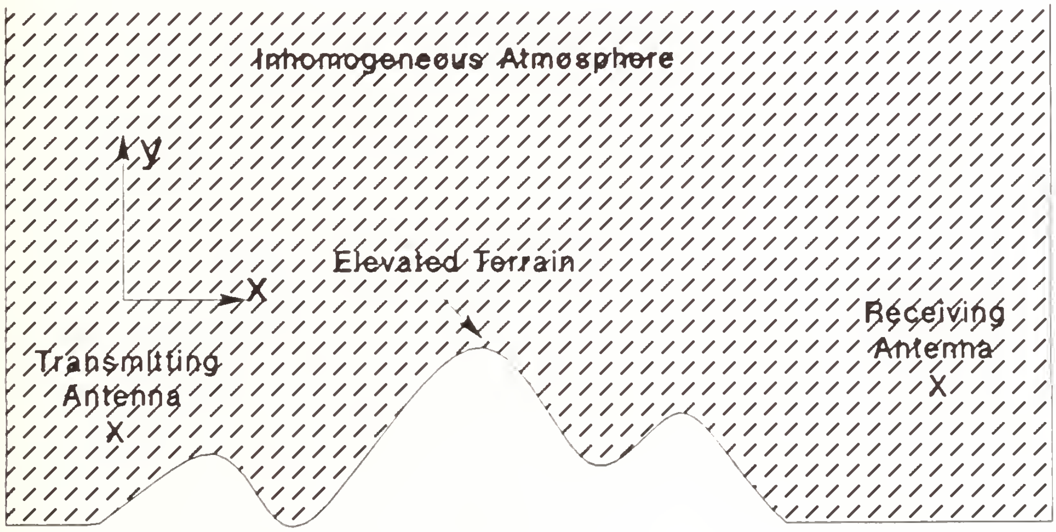
The method we propose to use will attempt to remove both of these defficiencies. Firstly, we use a Helmholtz-like elliptic equation to describe the fields and arrive at a wide-angle parabolic equation subject to certain approximations. To facilitate imposition of the boundary conditions on the irregular terrain, the equations will be transformed to a body-fitted curvilinear coordinate system. The PE will be solved using finite differences on a *non-rectangular* mesh. This is a major departure from previous approaches which will not only make the method more efficient but also more accurate.

In Section 2, we derive the exact equations satisfied by 2D fields in an inhomogeneous atmosphere. Impedance boundary conditions are used to characterize the ground. In Section 3, we present details on the impedance boundary conditions. Starting from the exact equations presented in Section 2 for the fields, we derive, in Section 4, a parabolic equation (PE) valid for narrow angle propagation. This case is termed as the standard PE. The standard PE is generally valid for propagation angles that are within $\pm 10°$ from horizontal. To accomodate waves traveling at larger angles, we present the derivation of a wide angle PE in Section 5. To truncate the computational domain we derive boundary conditions on an upper boundary, which are termed as the tropospheric boundary conditions. The derivation is based on the solution of Schrödinger type parabolic equation in a quarter plane $x > 0$, $y > 0$. This is presented in Section 6.

For an efficient numerical implementation, we transform the differential equation and boundary conditions to a curvilinear coordinate system. This is presented in Section 7. Details on the numerical generation of the curvilinear coordinate system are given in Section 8. Finally, in Section 9, we present steps leading to a finite difference solution of the equations using a Crank-Nicholson implicit scheme.

2. Solution of 2D Fields in an Inhomogeneous Medium

Consider an electric source producing fields in an inhomogeneous region as shown in the figure below. Let us assume that both the sources and the medium are two-dimensional in nature in that all quantities are independent of the z-coordinate. As in the case of a homogeneous medium the fields can be decomposed into a TE_z case (vertical polarization) and a TM_z case (horizontal polarization). It is assumed that propagation takes place in the xy-plane.



From Maxwell's equations, we have

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{1}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} + \vec{J} \tag{2}$$

In the case of vertical polarization, the fields could be written in terms of the z-component of the magnetic field, $\vec{H} = \hat{z}H_z$, (TE_z fields). Substituting into (2) we have

$$\nabla \times \vec{H} = \nabla \times (\hat{z}H_z) = \nabla H_z \times \hat{z} = j\omega\epsilon\vec{E} + \vec{J} \Longrightarrow \epsilon\vec{E} = \frac{\nabla H_z \times \hat{z} - \vec{J}}{j\omega}$$

In a source-free environment, we have

$$\epsilon\vec{E} = \frac{\nabla H_z \times \hat{z}}{j\omega}$$

Substituting into (1) we get

$$\nabla \times \vec{E} = \frac{1}{j\omega}\nabla \times \left(\frac{\nabla H_z}{\epsilon} \times \hat{z}\right) = -\frac{\hat{z}}{j\omega}\nabla \cdot \left(\frac{\nabla H_z}{\epsilon}\right) = -j\omega\mu\vec{H}$$

The equation satisfied by the magnetic field is then

$$-\frac{1}{j\omega}\nabla\cdot\left(\frac{\nabla H_z}{\epsilon}\right) = -j\omega\mu H_z \Longrightarrow$$

$$\boxed{\nabla\cdot\left(\frac{\nabla H_z}{\epsilon}\right) + \omega^2\mu H_z = 0} \qquad \text{Vertical Polarization} \tag{3}$$

Once the magnetic field is determined the electric field is given by

$$\boxed{\vec{E} = \frac{\hat{z}\times\nabla H_z}{j\omega\epsilon}} \tag{4}$$

Note that H_z does not satisfy the Helmholtz equation unless ϵ is constant.

For horizontal polarization on a similar analysis with $\vec{E} = \hat{z}E_z$ shows that

$$\boxed{\nabla\cdot\left(\frac{\nabla E_z}{\mu}\right) + \omega^2\epsilon E_z = 0} \qquad \text{Horizontal Polarization} \tag{5}$$

If the medium is non-magnetic, $\mu = \mu_0$ and E_z satisfies the Helmholtz equation. We will characterize the ground in terms of impedance boundary conditions [3].

We may combine the vertical and horizontal cases shown in (3) and (5) into an equation of the form

$$\nabla\cdot(\alpha\nabla\psi) + \beta\psi = 0 \tag{6}$$

where

$$\alpha = \begin{cases} \dfrac{1}{\epsilon} & \text{TE or Vertical Pol.} \\[2ex] \dfrac{1}{\mu} & \text{TM or Horizontal Pol.} \end{cases} \tag{7}$$

$$\beta = \alpha k_0^2 n^2(x,y) > 0 \tag{8}$$

$$\psi = \begin{cases} H_z & \text{TE Pol.} \\[1ex] E_z & \text{TM Pol.} \end{cases} \tag{9}$$

The quantity $n(x,y)$ is a position dependent refractive index of the medium. The partial differential equation (PDE) given in (6) is elliptic, for we have

$$\frac{\partial}{\partial x}\left(\alpha\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha\frac{\partial\psi}{\partial y}\right) + \beta\psi = 0$$

or

$$\alpha\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]+\frac{\partial\psi}{\partial x}\frac{\partial\alpha}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\alpha}{\partial y}+\beta\psi=0$$

The discriminant for the above PDE is $-1<0$ implying elliptic nature of the equation. Equation (6) could be expressed as

$$\nabla^2\psi+\frac{1}{\alpha}\frac{\partial\alpha}{\partial x}\psi_x+\frac{1}{\alpha}\frac{\partial\alpha}{\partial y}\psi_y+\frac{\beta}{\alpha}\psi=0 \tag{10}$$

For a non-magnetic, loss-less medium, we have

$$\alpha=\begin{cases}\dfrac{1}{\epsilon_0\epsilon_r}, & \text{Vertical Pol.}\\[2ex] \dfrac{1}{\mu_0}, & \text{Horizontal Pol.}\end{cases}$$

so that for vertical polarization

$$\begin{aligned}\frac{1}{\alpha}\frac{\partial\alpha}{\partial x} &= \epsilon_r\frac{\partial}{\partial x}\left(\frac{1}{\epsilon_r}\right)=-\frac{1}{\epsilon_r}\frac{\partial\epsilon_r}{\partial x}=-\frac{1}{n^2}\frac{\partial n^2}{\partial x}\\ &= -\frac{2}{n}\frac{\partial n}{\partial x}\quad (n\text{ is the refractive index})\end{aligned}$$

Similarly,

$$\frac{1}{\alpha}\frac{\partial\alpha}{\partial y}=-\frac{2}{n}\frac{\partial n}{\partial y}$$

Letting

$$a_1(x,y)=\begin{cases}-\dfrac{2}{n}\dfrac{\partial n}{\partial x} & \text{Vertical Pol.}\\[2ex] 0 & \text{Horizontal Pol.}\end{cases}$$

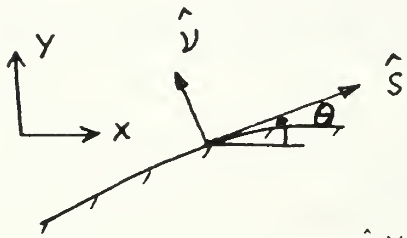
$$a_2(x,y)=\begin{cases}-\dfrac{2}{n}\dfrac{\partial n}{\partial y} & \text{Vertical Pol.}\\[2ex] 0 & \text{Horizontal Pol.}\end{cases}$$

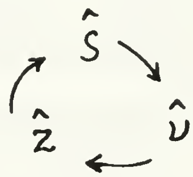
equation (10) may be expressed in the form

$$\nabla^2\psi+a_1\psi_x+a_2\psi_y+k_0^2n^2\psi=0 \tag{11}$$

3. Impedance Boundary Condition

Impedance boundary condition relates the tangential components of electric and magnetic fields at the interface of two media. If $\hat{\nu}$ is a unit normal and $\hat{s}$ is a unit tangent as shown in the figure below, the boundary conditions are given by [3]





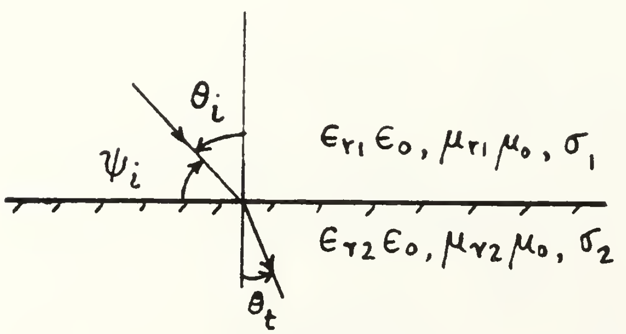
$$\hat{s} = \hat{x}\cos\theta + \hat{y}\sin\theta$$
$$\hat{\nu} = -\hat{x}\sin\theta + \hat{y}\cos\theta$$
$$\hat{x} = \hat{s}\cos\theta - \hat{\nu}\sin\theta$$
$$\hat{y} = \hat{s}\cos\theta + \hat{\nu}\cos\theta$$

$$\hat{\nu} \times (\hat{\nu} \times \vec{E}) = -\eta_0 \Delta_s \hat{\nu} \times \vec{H} \tag{12}$$

where $\Delta_s = Z_s/\eta_0$ is the surface impedance normalized to the free space value η_0. The equation may also be expressed as

$$\begin{aligned}(\hat{\nu}\cdot\vec{E})\hat{\nu} - \vec{E} = -\eta_0\Delta_s\hat{\nu}\times\vec{H} \Longrightarrow \hat{\nu}\times\vec{E} &= \eta_0\Delta_s\hat{\nu}\times(\hat{\nu}\times\vec{H}) \\ &= \eta_0\Delta_s\left[(\hat{\nu}\cdot\vec{H})\hat{\nu} - \vec{H}\right] \end{aligned} \tag{13}$$

The surface impedance is determined from the intrinsic impedance of the medium by considering plane wave reflections from the interface. The complex propagation constants, γ_1, γ_2, and the intrinsic impedances η_1 and η_2 in terms of the media constants are indicated in the figure.



$$\begin{aligned}\gamma_1^2 &= j\omega\mu_{r1}\mu_0(\sigma_1 + j\omega\epsilon_0\epsilon_{r1}) \\ &= -k_0^2\mu_{r1}\left(\epsilon_{r1} - j\frac{\sigma_1}{\omega\epsilon_0}\right) \\ &= -k_0^2\mu_{r1}\epsilon_{rc1} \\ \gamma_2^2 &= -k_0^2\mu_{r2}\epsilon_{rc2}\end{aligned}$$

$$\eta_1 = \sqrt{\frac{j\omega\mu_0\mu_{r1}}{\sigma_1 + j\omega\epsilon_0\epsilon_{r1}}} = \eta_0\sqrt{\frac{\mu_{r1}}{\epsilon_{rc1}}}, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$\eta_2 = \eta_0\sqrt{\frac{\mu_{r2}}{\epsilon_{r2}}}$$

According to Snell's law, we have $\gamma_1\sin\theta_i = \gamma_2\sin\theta_t$

The plane wave reflection coefficients for the vertical and horizontal polarizations, R_V and R_H are [7]

$$
\begin{aligned}
R_H &= \frac{\eta_2 \sec\theta_t - \eta_1 \sec\theta_i}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_2} \Longrightarrow \text{Surface Impedance } Z_s^H = \eta_2 \sec\theta_t \\
R_V &= \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i} \Longrightarrow Z_s^V = \eta_2 \cos\theta_t \\
\therefore \quad \Delta_s^H &= \frac{\eta_2 \sec\theta_t}{\eta_1} = \frac{\eta_2}{\eta_1}\left[1 - \left(\frac{\gamma_1}{\gamma_2}\sin\theta_i\right)^2\right]^{-1/2} \\
&= \sqrt{\frac{\mu_{r2}\epsilon_{rc1}}{\mu_{r1}\epsilon_{rc2}}} \cdot \left[1 - \frac{\mu_{r1}\epsilon_{rc1}}{\mu_{r2}\epsilon_{rc2}}\cos^2\psi_i\right]^{-1/2}
\end{aligned}
$$

and

$$\Delta_s^V = \sqrt{\frac{\mu_{r2}\epsilon_{rc1}}{\mu_{r1}\epsilon_{rc2}}} \cdot \left[1 - \frac{\mu_{r1}\epsilon_{rc1}}{\mu_{r2}\epsilon_{rc2}}\cos^2\psi_i\right]^{1/2}$$

For the special case of $\mu_1 = \mu_2 = \mu_0$, $\sigma_1 = 0$,

$$\Delta_s^H = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2} - j\sigma_{r2}}} \cdot \left[1 - \frac{\epsilon_{r1}}{\epsilon_{r2} - j\sigma_{r2}}\cos^2\psi_i\right]^{-1/2} \tag{14}$$

$$\Delta_s^V = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2} - j\sigma_{r2}}} \cdot \left[1 - \frac{\epsilon_{r1}}{\epsilon_{r2} - j\sigma_{r2}}\cos^2\psi_i\right]^{1/2} \tag{15}$$

For normal incidence $\psi_i = 90°$

$$\Delta_s^H = \Delta_s^V = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2} - j\sigma_{r2}}}$$

For the 2-D case, impedance boundary conditions for vertical polarization can be simplified as

$$\hat{\nu} \times \vec{E} = \eta_0 \Delta_s^V \left[(\hat{\nu} \cdot \vec{H})\hat{\nu} - \vec{H}\right] = -\eta_0 \Delta_s^V \vec{H}$$

Taking a dot product on both sides with $\hat{z}$

$$\hat{z} \cdot (\hat{\nu} \times \vec{E}) = -\eta_0 \Delta_s^V H_z$$

Since $z \times \hat{\nu} = -\hat{s}$, we have

$$-\hat{s} \cdot \vec{E} = -\eta_0 \Delta_s^V H_z$$

Substituting from equation (4) for $\vec{E}$, we get

$$\hat{s} \cdot \frac{\hat{z} \times \nabla H_z}{j\omega\epsilon} = -\eta_0 \Delta_s^v H_z$$

$$\hat{\nu} \cdot \nabla H_z = j\omega\epsilon\eta_0\Delta_s^V H_z = jk_0\epsilon_r\Delta_s^V H_z$$

i.e., the above can be put conveniently in the form

$$\boxed{\frac{\partial H_z}{\partial \nu} - jk_0\epsilon_r\Delta_s^V H_z = 0} \qquad \text{IBC Vertical Pol.} \tag{16}$$

A similar analysis for horizontal polarization yields

$$\boxed{\frac{\partial E_z}{\partial \nu} - jk_0\mu_r\frac{1}{\Delta_s^H}E_z = 0} \qquad \text{IBC Horizontal Pol.} \tag{17}$$

This may also be obtained by resorting to duality.

For a perfectly conducting material, we have $\Delta_s^{H/V} = 0$ and

$$\begin{aligned} \frac{\partial H_z}{\partial \nu} &= 0 \qquad \text{Vertical Pol.} \\ E_z &= 0 \qquad \text{Horizontal Pol.} \end{aligned}$$

4. Standard PE Derivation

We will make some approximations and cast (11) in the form of a parabolic equation which permits a rapid numerical solution.

Let $\psi(x,y) = \dfrac{e^{-jk_0 x}}{\sqrt{x}} u(x,y)$

Then

$$\psi_x = \left(u_x - jk_0 u - \frac{u}{2x^{3/2}}\right)\frac{e^{-jk_0 x}}{\sqrt{x}} \sim (u_x - jk_0 u)\frac{e^{-jk_0 x}}{\sqrt{x}}, \quad x \to \infty$$

$$\psi_y = u_y \frac{e^{-jk_0 x}}{\sqrt{x}}, \quad u_{yx} = u_{xy} \sim (u_{xy} - jk_0 u_y)\frac{e^{-jk_0 x}}{\sqrt{x}}$$

$$\psi_{yy} = u_{yy} \frac{e^{-jk_0 x}}{\sqrt{x}}, \quad u_{xx} \sim (u_{xx} - 2jk_0 u_x - k_0^2 u)\frac{e^{-jk_0 x}}{\sqrt{x}}$$

Substituting into (17) we get

$$u_{xx} + u_{yy} - 2jk_0 u_x + a_1 u_x - 2jk_0 a_1 u + a_2 u_y + (n^2 - 1)k_0^2 u = 0$$

or

$$u_{xx} + u_{yy} + (a_1 - 2jk_0)u_x + a_2 u_y + \left(n^2 - 1 - 2j\frac{a_1}{k_0}\right) k_0^2 u = 0$$

If now we impose the approximation that

$$|u_{xx}| \ll \left(a_1^2 + 4k_0^2\right)^{1/2} |u_x| \,,$$

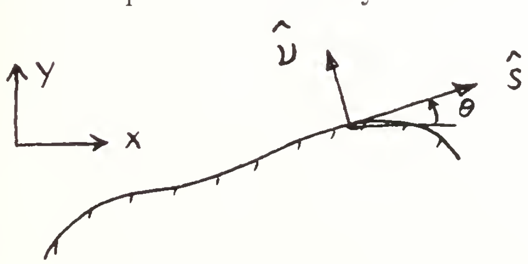
we obtain

$$u_x = \frac{-1}{(a_1 - 2jk_0)}\left\{ u_{yy} + a_2 u_y + \underbrace{\left(n^2 - 1 - 2j\frac{a_1}{k_0}\right) k_0^2\, u}_{a_1^*} \right\}$$

or

$$\boxed{u(x) = \frac{-j}{(2k_0 + ja_1)}\left\{ a_1^* + a_2\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right\} u} \qquad (18)$$

This is the exact form of narrow angle PE approximation. We would also like to express the impedance boundary condition in terms of the 'u' functions.



$$\begin{aligned} \hat{x} &= \hat{s}\cos\theta - \hat{\nu}\sin\theta \\ x_\nu = \frac{\partial x}{\partial \nu} &= \hat{\nu}\cdot\nabla x = \hat{\nu}\cdot\hat{x} \\ &= -\sin\theta \\ \hat{\nu} &= -\hat{x}\sin\theta + \hat{y}\cos\theta \end{aligned}$$

For vertical polarization we have

$$\frac{\partial H_z}{\partial \nu} - jk_0\epsilon_r\Delta_s^V H_z = 0 \Longrightarrow \frac{\partial H_z}{\partial \nu} - jk_0 n^2\Delta_s^V H_z = 0$$

$$\begin{aligned} H_z(x,y) &= \frac{e^{-jk_0x}}{\sqrt{x}}u(x,y) \\ \frac{\partial H_z}{\partial \nu} &= \left(u_\nu - jk_0x_\nu u - \frac{ux_\nu}{2x^3/2}\right)\frac{e^{-jk_0x}}{\sqrt{x}} \\ &\sim (u_\nu - jk_0x_\nu u)\frac{e^{-jk_0x}}{\sqrt{x}}, \qquad x \to \infty \end{aligned}$$

$$\therefore \boxed{u_\nu - jk_0(n^2\Delta_s^V + x_\nu)u = 0} \quad \text{IBC Vert. Pol.}$$

Similarly for horizontal polarization, we have

$$\boxed{u_\nu - jk_0\left(\frac{1}{\Delta_s^H} + x_\nu\right)u = 0} \quad \text{IBC Horz. Pol.}$$

We combine the two by defining

$$c_1 = \begin{cases} -jk_0(n^2\Delta_s^V - \sin\theta) & \text{Vert.} \\ -jk_0\left(\dfrac{1}{\Delta_s^H} - \sin\theta\right) & \text{Horz.} \end{cases}$$

and writing as

$$u_\nu + c_1u = 0 \tag{19}$$

The parabolic equation given in (18) is valid for propagating angles close to horizontal ($\pm 10°$ in practice) [1]. To accomodate waves at higher angles we would need a wide angle parabolic equation whose derivation is accomplished through a pseudo-differential operator formalism [1].

5. Wide Angle PE Derivation

Let us assume that the media constants are independent of horizontal range so that $\alpha(x,y) = \alpha(y)$, $n(x,y) = n(y)$. We then have from (17)

$$u_{xx} + u_{yy} - j2k_0 u_x + \frac{1}{\alpha}\frac{d\alpha}{dy}u_y + (n^2(y) - 1)k_0^2 u = 0$$

Let

$$P = \frac{\partial}{\partial x}, \quad Q = \sqrt{\frac{1}{k_0^2}\frac{\partial^2}{\partial y^2} + \frac{1}{k_0^2\alpha}\frac{d\alpha}{dy}\frac{\partial}{\partial y} + n^2} \quad \left(\begin{array}{l}\text{Pseudo differential} \\ \text{operator in } y\end{array}\right)$$

Using this notation, the above PDE may be expressed as

$$\left[P^2 - 2jk_0 P + (Q^2 - 1)k_0^2\right] u = 0$$

which we may factorize as

$$(P - jk_0 - jk_0 Q)(P - jk_0 + jk_0 Q)u = 0 \tag{20}$$

To see this we expand the operator on the left hand side to get

$$\begin{aligned} P^2 - jk_0 P \;&-\; jk_0 QP - jk_0 P - k_0^2 - k_0^2 Q \\ &+\; jk_0 PQ + k_0^2 Q^2 + k_0^2 Q \end{aligned}$$

Since $PQ = QP$, we have the desired result. The first operator in (20) denotes an incoming wave (w.r.t. x) and the second an outgoing wave. We retain only the outgoing wave to obtain

$$Pu = -jk_0(Q - 1)u \tag{21}$$

The square root operator Q is global in nature and we would like to make some approximations to derive a local operator from it. (It is global because when expanded in terms of series, it will contain terms of all derivatives). Now

$$Q = \left[n^2 + \frac{1}{k_0^2}\frac{1}{\alpha}\frac{d\alpha}{dy}\frac{\partial}{\partial y} + \frac{1}{k_0^2}\frac{\partial^2}{\partial y^2}\right]^{1/2}$$

Let us rewrite Q as

$$Q = \left[1 + \underbrace{[n^2(y) - 1]}_{\ll 1 \text{ normally}} + \underbrace{\frac{1}{\alpha}\frac{d\alpha}{d(k_0 y)}\frac{\partial}{\partial(k_0 y)} + \frac{\partial^2}{\partial(k_0 y)^2}}_{\text{small}}\right]^{1/2}$$

which may be expressed as

$$Q = \sqrt{1+V}$$

where

$$\begin{aligned} V &= n^2 - 1 + \frac{1}{k_0^2\alpha}\frac{d\alpha}{dy}\frac{\partial}{\partial y} + \frac{1}{k_0^2}\frac{\partial^2}{\partial y^2} \\ &= \text{a small operator!} \end{aligned}$$

Treating V as an algebraic factor < 1, we may derive the following rational approximation ($pade(1,1)$)

$$\begin{aligned} Q = \sqrt{1+V} &\approx \frac{1+\frac{3}{4}V}{1+\frac{1}{4}V} \quad \text{(Claerbout)} \\ &= \frac{4+3V}{4+V} = 1 + \frac{2V}{4+V} \end{aligned}$$

so that

$$Q - 1 = \left(\frac{2V}{4+V}\right)$$

Substituting into (21) we arrive at

$$Pu = jk_0(Q-1)u \Longrightarrow Pu = \frac{-2jk_0V}{4+V}u$$

or

$$(4+V)Pu = -2jk_0Vu$$

Using the fact that $a_2 = \frac{1}{\alpha}\frac{d\alpha}{dy}$, we get

$$\left[(n^2+3)k_0^2 + a_2\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right]u_x = -2jk_0\left[(n^2-1)k_0^2 + a_2\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right]u \qquad (22)$$

The above equation is a wide-angle parabolic equation valid for propagation up to $\pm 20°$ [1]. Other approximations could be obtained by considering higher order *pade* approximants.

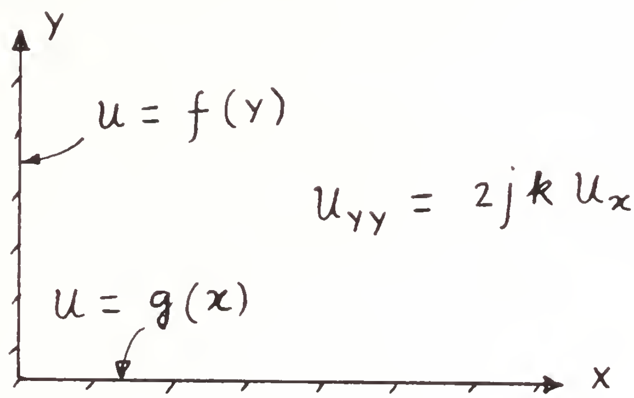
6. Boundary Condition on the Upper Boundary

To truncate the computational domain, we consider a point high enough where the atmosphere is homogeneous with $n = 1$. The governing equation in the homogeneous region becomes

$$u_x = -\frac{j}{2k_0} u_{yy}$$

Let us derive boundary conditions on a horizontal interface $y = y_0$. For the sake of simplicity we will derive boundary conditions of the mixed type on the interface $y = 0$ (instead of $y = y_0$) given the initial data on the line $x = 0$ and boundary data on the line $y = 0$. Our derivation is based on the use of Fourier sine transforms as suggested in [2]. Although the basic philosophy of our approach coincides with that in [4], some of the details and the final results are slightly different from the latter.

Consider the parabolic equation $u_{yy} - 2jku_x = 0$ in a homogeneous region $x > 0$, $y > 0$, where k is a complex constant,



subject to the initial condition $u(0, y) = f(y)$, $0 < y < \infty$, and the boundary condition $u(x, 0) = g(x)$, $0 < x < \infty$. We assume that $u(x, \infty) \longrightarrow 0$ and $u_y(x, \infty) \longrightarrow 0$. The equation

$$u_{yy} = 2jku_x \tag{23}$$

is of Schrödinger's type. We will treat the lossless case having a real value of k as the limiting case of the lossy problem having $k = k_0 - j\epsilon$, $\epsilon > 0$. We will solve the problem using Fourier sine integral.

Let

$$U_s(x, \lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \sin \lambda y \, dy \tag{24}$$

Then using integration by parts, we see that

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\int_0^\infty u_{yy}(x,y)\sin\lambda y\,dy &= \sqrt{\frac{2}{\pi}}\left\{u_y(x,y)\sin\lambda y\Big|_{y=0}^{\infty}\right.\\ &\left.-\lambda u(x,y)\cos\lambda y\Big|_{y=0}^{\infty} - \lambda^2\int_0^\infty u(x,y)\sin\lambda y\,dy\right\}\end{aligned}$$

Because $u_y(x,\infty)\longrightarrow 0$ and $u(x,\infty)\longrightarrow 0$, we have

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\int_0^\infty u_{yy}(x,y)\sin\lambda u\,dy &= \sqrt{\frac{2}{\pi}}\lambda u(x,0) - \lambda^2 U_s(x,\lambda)\\ &= \sqrt{\frac{2}{\pi}}\lambda g(x) - \lambda^2 U_s(x,\lambda) \qquad (25)\end{aligned}$$

Multiplying both sides of (23) with $\sqrt{2/\pi}\sin\lambda y$, integrating over $y = 0$ to ∞ and making use of (25), we have

$$\sqrt{\frac{2}{\pi}}\lambda g(x) - \lambda^2 U_s(x,\lambda) = 2jk\frac{\partial}{\partial x}U_s(x,\lambda)$$

or

$$\frac{\partial}{\partial x}U_s(x,\lambda) + \frac{\lambda^2}{2jk}U_s(x,\lambda) = \frac{\lambda}{2jk}\sqrt{\frac{2}{\pi}}g(x)$$

We may rewrite the above equation as

$$\frac{\partial}{\partial x}\left[U_s(x,\lambda)e^{(\lambda^2/2jk)x}\right] = \frac{\lambda}{2jk}\sqrt{\frac{2}{\pi}}g(x)e^{(\lambda^2/2jk)x} \qquad (26)$$

Replacing the dummy variable x in (26) with τ and integrating both sides over $\tau = 0$ to x, we arrive at

$$U_s(\tau,\lambda)e^{(\lambda^2/2jk)\tau}\Big|_{\tau=0}^{x} = \frac{\lambda}{2jk}\sqrt{\frac{2}{\pi}}\int_{\tau=0}^{x} g(\tau)e^{(\lambda^2/2jk)\tau}d\tau$$

or

$$U_s(x,\lambda) = U_s(0,\lambda)e^{-(\lambda^2/2jk)x} + \frac{\lambda}{2jk}\sqrt{\frac{2}{\pi}}\int_{\tau=0}^{x} g(\tau)e^{(\lambda^2/2jk)(\tau-x)}d\tau$$

But

$$\begin{aligned}U_s(0,\lambda) &= \sqrt{\frac{2}{\pi}}\int_0^\infty u(0,y)\sin\lambda y\,dy\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\sin\lambda y\,dy = F_s(\lambda)\end{aligned}$$

$$\therefore \qquad U_s(x,\lambda) = F_s(\lambda)e^{-(\lambda^2/2jk)x} + \frac{\lambda}{2jk}\sqrt{\frac{2}{\pi}}\int_{\tau=0}^{x} g(\tau)e^{(\lambda^2/2jk)(\tau-x)}d\tau \tag{27}$$

Finally, taking the inverse sine transform on (27) we get

$$\begin{aligned}
u(x,y) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin\lambda y\, U_s(x,\lambda)d\lambda \\
&= \sqrt{\frac{2}{\pi}}\int_{\lambda=0}^\infty F_s(\lambda)e^{-(\lambda^2/2jk)x}\sin\lambda y\, d\lambda \\
&\quad + \frac{2}{\pi}\frac{1}{2jk}\int_{\lambda=0}^\infty \lambda\sin\lambda y\int_{\tau=0}^{x} g(\tau)e^{(\lambda^2/2jk)(\tau-x)}d\tau\, d\lambda \\
&= \frac{2}{\pi}\int_{\lambda=0}^\infty\int_{\tau=0}^\infty f(\tau)\sin\lambda\tau\, d\tau e^{-(\lambda^2/2jk)x}\sin\lambda y\, d\lambda \\
&\quad + \frac{2}{\pi}\frac{1}{2jk}\int_{\tau=0}^{x} g(\tau)\int_{\lambda=0}^\infty \lambda\sin\lambda y\, e^{(\lambda^2/2jk)(\tau-x)}d\lambda\, d\tau \\
&= \frac{1}{\pi}\int_{\tau=0}^\infty f(\tau)\int_{\lambda=0}^\infty [\cos(\tau-y)\lambda-\cos(\tau+y)\lambda]\, e^{-(\lambda^2/2jk)x}d\lambda\, d\tau \\
&\quad + \frac{1}{\pi}\int_{\tau=0}^{x} g(\tau)\frac{1}{jk}\left(-\frac{\partial}{\partial y}\right)\int_{\lambda=0}^\infty \cos\lambda y e^{(\lambda^2/2jk)(\tau-x)}d\lambda\, d\tau \\
&= \frac{1}{\pi}\int_{\tau=0}^\infty f(\tau)\int_{\lambda=0}^\infty [\cos(y-\tau)\lambda-\cos(y+\tau)\lambda]\, e^{-(\lambda^2/2jk)x}d\lambda\, d\tau \\
&\quad - \frac{1}{\pi jk}\int_{\tau=0}^{x} g(\tau)\frac{\partial}{\partial y}\left\{\int_0^\infty \cos y\lambda\, e^{-(\lambda^2/2jk)(x-\tau)}d\lambda\right\}d\tau
\end{aligned}$$

Defining

$$K(x,y;\,x_0,y_0) = \int_0^\infty \cos(y-y_0)\lambda e^{-(\lambda^2/2jk)(x-x_0)}d\lambda, \quad \text{for } x>x_0,$$

we write the expression for $u(x,y)$ as

$$\begin{aligned}
u(x,y) &= \frac{1}{\pi}\int_{\tau=0}^\infty f(\tau)\,[K(x,y;\,0,\tau)-K(x,y;\,0,-\tau)]\,d\tau \\
&\quad - \frac{1}{\pi jk}\int_{\tau=0}^{x} g(\tau)\frac{\partial}{\partial y}K(x,y;\tau,0)d\tau
\end{aligned} \tag{28}$$

We now evaluate the integral for K. Consider

$$I(\alpha) = \int_{\lambda=0}^\infty \cos\alpha\lambda e^{-(\lambda^2/2jk)(x-x_0)}d\lambda\,, \qquad x>x_0$$

$$= \int_{\lambda=0}^{\infty} \cos\alpha\lambda e^{(\lambda^2/2|k|^2)jk^*(x-x_0)} d\lambda$$

$$= \int_{\lambda=0}^{\infty} \cos\alpha\lambda e^{-(\lambda^2(x-x_0)/2|k|^2)(\epsilon-jk_0)} d\lambda$$

Because of the exponential decay, we may differentiate under the integral sign to obtain

$$\begin{aligned}
\frac{d}{d\alpha}I(\alpha) &= -\int_{\lambda=0}^{\infty} \lambda \sin\alpha\lambda e^{-(\lambda^2(x-x_0)/2|k|^2)(\epsilon-jk_0)} d\lambda \\
&= \int_{\lambda=0}^{\infty} \sin\alpha\lambda \frac{\frac{\partial}{\partial\lambda}\left[e^{-(\lambda^2(x-x_0)/2|k|^2)(\epsilon-jk_0)}\right]}{(x-x_0)(\epsilon-jk_0)/|k|^2} d\lambda \\
&= \left.\frac{\sin\alpha\lambda e^{-\lambda^2(x-x_0)(\epsilon-jk_0)/2|k|^2}}{(x-x_0)(\epsilon-jk_0)/|k|^2}\right|_{\lambda=0}^{\infty} - \frac{\alpha|k|^2}{(x-x_0)(\epsilon-jk_0)} \\
&\quad \int_0^{\infty} \cos\alpha\lambda e^{-(\lambda^2(x-x_0)(\epsilon-jk_0))/2|k|^2} d\lambda \\
&= -\frac{j\alpha|k|^2}{(x-x_0)k^*}\int_0^{\infty} \cos\alpha y e^{-\lambda^2(x-x_0)/2jk} d\lambda = -\frac{j\alpha k}{(x-x_0)}I(\alpha)
\end{aligned}$$

$$\therefore \qquad \frac{dI(\alpha)}{d\alpha} + \frac{j\alpha k}{(x-x_0)}I(\alpha) = 0$$

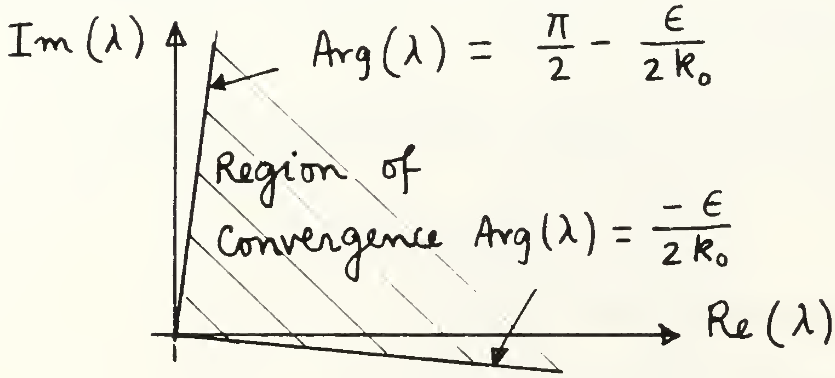
or

$$\frac{d}{d\alpha}\left[I(\alpha)e^{j\alpha^2k/(2(x-x_0))}\right] = 0 \Longrightarrow I(\alpha) = I(\alpha=0)e^{-j\alpha^2k/(2(x-x_0))} \tag{29}$$

Now $I(\alpha = 0)$ can be written as

$$I(\alpha=0) = \int_0^{\infty} e^{-(\lambda^2(x-x_0)/2|k|^2)(\epsilon-jk_0)} d\lambda$$

Let us view this integral in the complex λ-plane.



Complex λ-Plane

For the integral to converge for $(x - x_0) > 0$, $Re[\lambda^2(\epsilon - jk_0)] > 0$, i.e.,

$$Re[\lambda^2(-jk^*)] > 0 \Longrightarrow Im(\lambda^2 k^*) > 0$$

or

$$0 < \mathrm{Arg}\,(\lambda^2 k^*) < \pi$$

or

$$0 < 2\,\mathrm{Arg}\,(\lambda) - \mathrm{Arg}\,(k) < \pi$$

or

$$\frac{1}{2}\,\mathrm{Arg}\,(k) < \mathrm{Arg}\,(\lambda) < \frac{\pi}{2} + \frac{1}{2}\,\mathrm{Arg}\,(k)$$

Now

$$\mathrm{Arg}\,(k) = -\tan^{-1}\left(\frac{\epsilon}{k_0}\right) \approx -\frac{\epsilon}{k_0}, \qquad \frac{\epsilon}{k_0} \ll 1$$

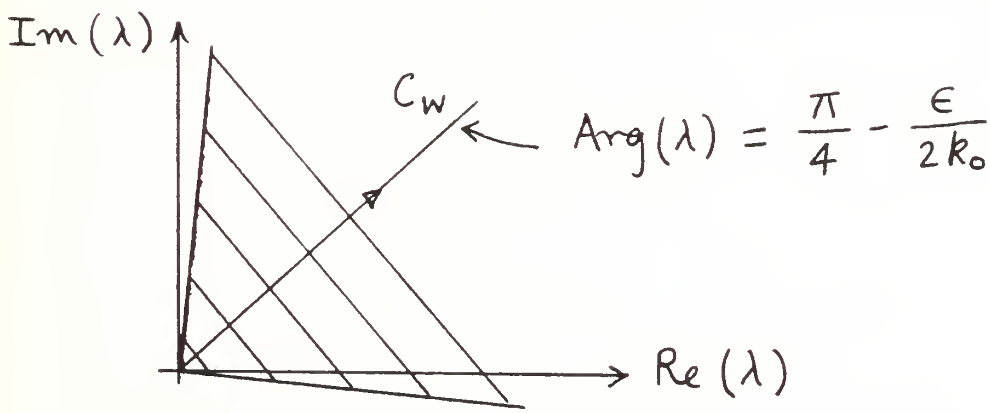
$$\therefore \qquad -\frac{\epsilon}{2k_0} < \mathrm{Arg}\,(\lambda) < \frac{\pi}{2} - \frac{\epsilon}{2k_0} \tag{30}$$

$$I(\alpha = 0) = \int_C e^{-\lambda^2(x-x_0)(-jk^*)/2|k|^2}\, d\lambda$$

where C lies in the region of convergence.

In particular, let us choose a line from 0 to ∞ along the line C_w defined by

$$\lambda = \sqrt{\frac{2jk}{x - x_0}}\,w \qquad w = 0 \text{ to } \infty \tag{31}$$



On this path,

$$\mathrm{Arg}\,(\lambda) = \frac{\pi}{4} + \frac{1}{2}\,\mathrm{Arg}\,(k) = \frac{\pi}{4} - \frac{\epsilon}{2k_0}$$

$$\therefore \qquad I(\alpha = 0) = \sqrt{\frac{2jk}{(x - x_0)}} \int_{w=0}^{\infty} e^{-w^2}\, dw = \sqrt{\frac{\pi jk}{2(x - x_0)}}$$

$$\therefore \qquad I(\alpha) = \sqrt{\frac{\pi j k}{2(x-x_0)}} e^{-j\alpha^2 k/(2(x-x_0))}$$

$$\therefore \qquad K(x,y;\; x_0,y_0) = \sqrt{\frac{\pi j k}{2(x-x_0)}} e^{-jk(y-y_0)^2/(2(x-x_0))} \tag{32}$$

Substituting this into (28), the field $u(x,y)$ for the initial-boundary value problem is given by

$$\begin{aligned} u(x,y) &= \frac{1}{\pi}\int_{\tau=0}^{\infty} f(\tau)\sqrt{\frac{\pi j k}{2x}}\left\{e^{-jk(y-\tau)^2/(2x)} - e^{-jk(y+\tau)^2/(2x)}\right\} d\tau \\ &\quad -\frac{1}{\pi j k}\int_{\tau=0}^{x} g(\tau)\frac{\partial}{\partial y}\left\{\sqrt{\frac{\pi j k}{2(x-\tau)}} e^{-jky^2/(2(x-\tau))}\right\} d\tau \end{aligned} \tag{33}$$

It is easy to see that

$$\begin{aligned} \frac{\partial K}{\partial x} &= \int_0^{\infty} -\frac{\lambda^2}{2jk}\cos[(y-y_0)\lambda] e^{-\lambda^2(x-\tau)/(2jk)} d\lambda \\ \frac{\partial^2 K}{\partial y^2} &= \int_0^{\infty} -\lambda^2 \cos[(y-y_0)\lambda] e^{-\lambda^2(x-\tau)/(2jk)} d\lambda \end{aligned}$$

and, so for $x \neq x_0$

$$\frac{\partial^2 K}{\partial y^2} - 2jk\frac{\partial K}{\partial x} = 0 \quad \text{or} \quad \frac{\partial K}{\partial x} = \frac{1}{2jk}\frac{\partial^2 K}{\partial y^2} \tag{34}$$

which is the same equation satisfied by u. Now

$$\begin{aligned} u(x,y) &= \sqrt{\frac{jk}{2\pi x}}\int_{\tau=0}^{\infty} f(\tau)\left[e^{-jk(y-\tau)^2/(2x)} - e^{-jk(y+\tau)^2/(2x)}\right] \\ &\quad -\frac{1}{\pi j k}\int_{\tau=0}^{x} g(\tau)\frac{\partial}{\partial y}K(x,y;\; \tau,0) d\tau \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial u}{\partial y}(x,y)\right|_{y\to 0^+} &= \sqrt{\frac{jk}{2\pi x}}\int_0^{\infty} f(\tau)\frac{2jk\tau}{x} e^{-jk\tau^2/(2x)} d\tau \\ &\quad -\frac{1}{\pi j k}\int_{\tau=0}^{x} g(\tau)\left.\frac{\partial^2}{\partial y^2}K(x,y;\; \tau,0)\right|_{y\to 0^+} d\tau \\ &= -\sqrt{\frac{jk}{2\pi x}}\int_0^{\infty} f(\tau) 2\frac{\partial}{\partial \tau} e^{-jk\tau^2/(2x)} d\tau \\ &\quad -\frac{1}{\pi j k}\int_0^{x} g(\tau) 2jk\left.\frac{\partial}{\partial x}K(x,y;\; \tau,0)\right|_{y\to 0^+} d\tau \end{aligned}$$

in view of (34). Furthermore, we note that

$$\frac{\partial}{\partial x}K(x,y;x_0,y_0) = -\frac{\partial}{\partial x_0}K(x_0y;x_0y_0)$$

Using this and evaluating the first integral by parts, we get

$$\begin{aligned}
\left.\frac{\partial}{\partial y}u(x,y)\right|_{y\to 0^+} &= -\sqrt{\frac{2jk}{\pi x}}\left\{\left. f(\tau)e^{-jk\tau^2/(2x)}\right|_{\tau=0}^{\infty} - \int_0^\infty f_\tau(\tau)e^{-jk\tau^2/(2x)}d\tau\right\} \\
&\quad +\frac{2}{\pi}\int_{\tau=0}^{x} g(\tau)\frac{\partial}{\partial\tau}K(x,y;\ \tau,0)\Big|_{y\to 0^+} d\tau \\
&= -\sqrt{\frac{2jk}{\pi x}}\left\{-f(0) - \int_0^\infty f_\tau(\tau)e^{-jk\tau^2/(2x)}d\tau\right\} \\
&\quad +\frac{2}{\pi}\left[g(\tau)k(x,y;\tau,0)\Big|_{\substack{\tau=0\\ y\to 0^+}} - \int_{\tau=0}^{x} g_\tau(\tau)K(x,0;\ \tau,0)d\tau\right] \\
&= +\sqrt{\frac{2jk}{\pi x}}f(0) + \sqrt{\frac{2jk}{\pi x}}\int_0^\infty f_\tau(\tau)e^{-jk\tau^2/(2x)}d\tau \\
&\quad -\frac{2}{\pi}g(0)\sqrt{\frac{\pi jk}{2x}} - \sqrt{\frac{2jk}{\pi}}\int_{\tau=0}^{x}\frac{g_\tau(\tau)}{\sqrt{x-\tau}}d\tau \\
&= \sqrt{\frac{2jk}{\pi x}}[f(0)-g(0)] \\
&\quad +\sqrt{\frac{2jk}{\pi}}\left\{\int_0^\infty \frac{f_\tau(\tau)}{\sqrt{x}}e^{-jk\tau^2/(2x)}d\tau - \int_0^x \frac{g_\tau(\tau)}{\sqrt{x-\tau}}d\tau\right\}
\end{aligned}$$

From the compatibility conditions on the initial and boundary values, we have

$$\lim_{x\to 0} u(x,0) = g(0) = \lim_{y\to 0} u(0,y) = f(0)$$

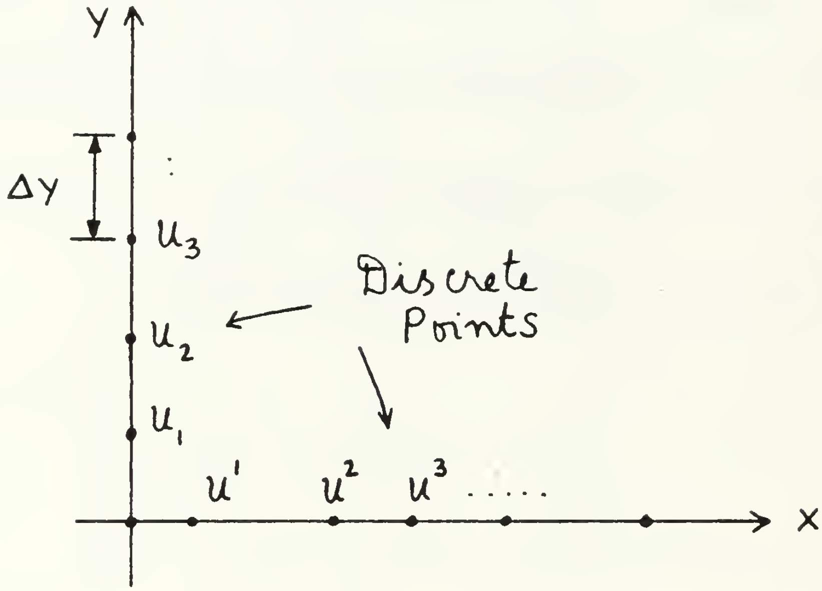
Therefore

$$\left.\frac{\partial u}{\partial y}(x,y)\right|_{y\to 0^+} = \sqrt{\frac{2jk}{\pi}}\left[\frac{1}{\sqrt{x}}\int_{\tau=0}^{\infty} f_\tau(\tau)e^{-jk\tau^2/(2x)}d\tau - \int_{\tau=0}^{x}\frac{g_\tau(\tau)}{\sqrt{x-\tau}}d\tau\right] \tag{35}$$

It is easy to note from (35) that for $k = k_0 - j\epsilon$,

$$\lim_{x\to 0}\left.\frac{\partial u}{\partial y}(x,y)\right|_{y\to 0^+} = 0.$$

No matter how small the ϵ is, we will use the same equality for the limiting case of $\epsilon \longrightarrow 0$. We will evaluate the integrals above approximately by replacing the derivatives with differences.



Consider initial data on the line $x = 0$. Let us assume that this initial data is known on a uniform grid $y_m = m\Delta y$, $m = 0, 1, \cdots$. We approximate the derivative in the interval (y_{m-1}, y_m) by the forward difference formula

$$\frac{\partial f(y)}{\partial y} \approx \frac{u_m - u_{m-1}}{\Delta y} \; ; \qquad y \in (y_{m-1}, y_m)$$

where $u_m = u(0, y_m)$.

Then

$$\frac{1}{\sqrt{x}} \int_{\tau=0}^{\infty} f_\tau(\tau) e^{-jk\tau^2/(2x)} d\tau \approx \sum_{m=1}^{\infty} \frac{u_m - u_{m-1}}{\Delta y}$$

$$\frac{1}{\sqrt{x}} \int_{\tau=y_{m-1}}^{y_m} e^{-jk\tau^2/(2x)} d\tau$$

Let

$$\sqrt{\frac{k}{x}}\tau = \sqrt{\pi}\,\mu \Longrightarrow \frac{1}{\sqrt{x}} d\tau = \sqrt{\frac{\pi}{k}} d\mu$$

We then have

$$\frac{1}{\sqrt{x}}\int_{\tau=y_{m-1}}^{y_m} e^{-j(k\tau^2)/2x}d\tau = \sqrt{\frac{\pi}{k}}\int_{\sqrt{k/(\pi x)}y_{m-1}}^{\sqrt{k/(\pi x)}y_m} e^{-j\pi\mu^2/2}d\mu$$

$$= \sqrt{\frac{\pi}{k}}F(\mu)\Big]_{\sqrt{k/(\pi x)}y_{m-1}}^{\sqrt{k/(\pi x)}y_m}$$

where $F(\mu)$ = complex Fresnel integral [8]

$$\therefore \quad \frac{1}{\sqrt{x}}\int_{\tau=0}^{\infty} f_\tau(\tau)e^{-jk\tau^2/(2x)}d\tau = \sum_{m=1}^{\infty}\left(\frac{u_m - u_{m-1}}{\Delta y}\right)\sqrt{\frac{\pi}{k}}$$

$$\left[F\left(\sqrt{\frac{k}{\pi x}}y_m\right) - F\left(\sqrt{\frac{k}{\pi x}}y_{m-1}\right)\right] \quad (36)$$

Consider now the boundary data on the line $y = 0$. Assume that we have a non-uniform grid $0, x_1, x_2, \cdots, x_M = x$. On the interval (x_{m-1}, x_m), we approximate the derivative as

$$g_\tau(\tau) = \frac{u^m - u^{m-1}}{x_m - x_{m-1}}$$

where $u^m = u(x_m, 0)$. At the origin we have $u^0 = u_0$.

We evaluate the second integral in (35) as

$$\int_0^x \frac{g_\tau(\tau)}{\sqrt{x-\tau}}d\tau = \sum_{m=1}^{M}\frac{u^m - u^{m-1}}{x_m - x_{m-1}}\int_{x_{m-1}}^{x_m}\frac{1}{\sqrt{x-\tau}}d\tau$$

$$= \sum_{m=1}^{M}\frac{u^m - u^{m-1}}{x_m - x_{m-1}}\left(-2\sqrt{x-\tau}\right)\Big|_{x_{m-1}}^{x_m}$$

$$= 2\sum_{m=1}^{M}\frac{u^m - u^{m-1}}{x_m - x_{m-1}}\left(\sqrt{x - x_{m-1}} - \sqrt{x - x_m}\right)$$

$$= 2\sum_{m=1}^{M}\frac{u^m - u^{m-1}}{\sqrt{x_M - x_{m-1}} + \sqrt{x_M - x_m}} \quad (37)$$

Using (36) and (37) in (35) we get

$$\frac{\partial u}{\partial y}(x,y)\Big|_{y=0^+} = \sqrt{\frac{2jk}{\pi}}\left[\sqrt{\frac{\pi}{k}}\sum_{m=1}^{\infty}\frac{u_m - u_{m-1}}{\Delta y}\left\{F\left(\sqrt{\frac{k}{\pi x_M}}y_m\right) - F\left(\sqrt{\frac{k}{\pi x_M}}y_{m-1}\right)\right\}\right.$$

$$\left. -2\sum_{m=1}^{M}\frac{u^m - u^{m-1}}{\sqrt{x_M - x_m} + \sqrt{x_M - x_{m-1}}}\right]$$

Extracting out the $m = M$ term from the latter and writing

$$\left.\frac{\partial u}{\partial y}(x,y)\right|_{y\to 0^+} = \frac{\partial u^M}{\partial y}$$

we have

$$\begin{aligned}\frac{\partial u^M}{\partial y} + \sqrt{\frac{8jk}{\pi}}\frac{u^M}{\sqrt{x_M - x_{M-1}}} &= \sqrt{2j}\sum_{m=1}^{\infty}\frac{u_m - u_{m-1}}{\Delta y} \\ &\quad \left[F\left(\sqrt{\frac{k}{\pi x_M}}y_m\right) - F\left(\sqrt{\frac{k}{\pi x_M}}y_{m-1}\right)\right] \\ &\quad -\sqrt{\frac{8jk}{\pi}}\sum_{m=1}^{M-1}\frac{u^m - u^{m-1}}{\sqrt{x_M - x_m} + \sqrt{x_M - x_{m-1}}} \qquad (38)\end{aligned}$$

The above equation is the discrete version of a continuous boundary condition of the form

$$\frac{\partial u}{\partial y} + r(x)u = s(x) \qquad (39)$$

where

$$r(x) = \sqrt{\frac{8jk}{\pi}}\frac{1}{\sqrt{x - x_{M-1}}} \qquad (40)$$

and

$$\begin{aligned}s(x) &= \sqrt{2j}\sum_{m=1}^{\infty}\frac{u_m - u_{m-1}}{\Delta y}\left[F\left(\sqrt{\frac{k}{\pi x}}y_m\right) - F\left(\sqrt{\frac{k}{\pi x}}y_{m-1}\right)\right] \\ &\quad -\sqrt{\frac{8jk}{\pi}}\sum_{m=1}^{M-1}\frac{u^m - u^{m-1}}{\sqrt{x - x_m} + \sqrt{x - x_{m-1}}}, \qquad (41)\end{aligned}$$

and

$$F(x) = \int_0^x e^{-j(\pi/2)\tau^2}\,d\tau \qquad (42)$$

is the complex Fresnel integral [8].

For efficient implementation, the PDE and the various boundary conditions will be transformed into a curvilinear coordinate system generated by setting the lower irregular boundary as a constant curve curvilinear coordinate.

7. Transformations to a Curvilinear Coordinate System

Consider the narrow angle parabolic equation with range dependent refractive index ($a_1 \neq 0$) given in (18)

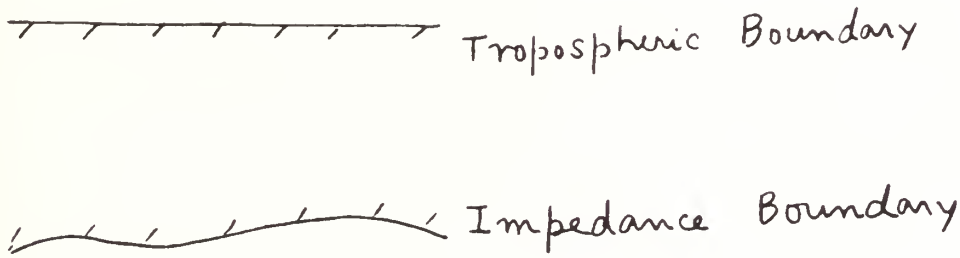
$$u_x = \frac{1}{(2jk_0 - a_1)} \left\{ a_1^* + a_2 \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} \right\} u \qquad \text{PDE}$$

together with the tropospheric boundary condition

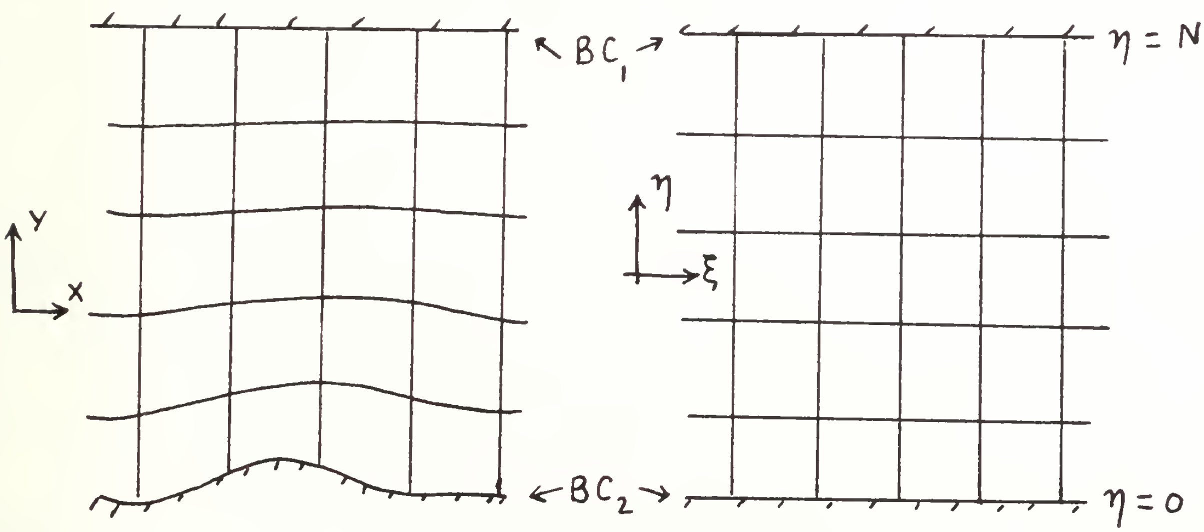
$$\frac{\partial}{\partial y} u(x_m, y_0) + r(x_m) u(x_m, y_0) = s(x_m, y_0)$$

and the impedance boundary condition on the irregular boundary

$$u_\nu + c_1 u = 0$$



We will transform these to a curvilinear coordinate system (ξ, η)



Physical Domain

Computational Domain

We assume that we have a transformation of the following form

$$x = x(\xi)\,, \qquad y = y(\xi, \eta)$$

The various metrics needed in the transformed equations are [5]

$$g_{11} = x_\xi^2 + y_\xi^2\,, \qquad g_{12} = x_\xi x_\eta + y_\xi y_\eta\,.$$

We then have with $\sqrt{g} = x_\xi y_\eta - x_\eta y_\xi = x_\xi y_\eta \neq 0$

$$\begin{aligned}
u_x &= \frac{1}{\sqrt{g}}(y_\eta u_\xi - y_\xi u_\eta) = \frac{1}{x_\xi y_\eta}(u_\xi y_\eta - u_\eta y_\xi) \\
u_y &= \frac{1}{\sqrt{g}}(-x_\eta u_\xi + x_\xi u_\eta) = \frac{u_\eta}{y_\eta} \\
u_{yy} &= \frac{1}{y_\eta}\frac{\partial}{\partial \eta}\left(\frac{u_\eta}{y_\eta}\right) = \frac{1}{y_\eta^2}\left[u_{\eta\eta} - \frac{y_{\eta\eta}}{y_\eta}u_\eta\right]
\end{aligned}$$

Substituting into the PDE we get

$$\frac{1}{x_\xi y_\eta}(u_\xi y_\eta - u_\eta y_\xi) = \frac{1}{(2jk_0 - a_1)}\left\{a_1^* u + a_2\frac{u_\eta}{y_\eta} - \frac{y_{\eta\eta}}{y_\eta^3}u_\eta + \frac{u_{\eta\eta}}{y_\eta^2}\right\}$$

or

$$u_\xi = \frac{x_\xi a_1^*}{(2jk_0 - a_1)}u + \left[\left(\frac{a_2}{y_\eta} - \frac{y_{\eta\eta}}{y_\eta^3}\right)\frac{x_\xi}{(2jk_0 - a_1)} + \frac{x_\xi y_\xi}{y_\eta}\right]u_\eta + \frac{x_\xi}{(2jk_0 - a_1)y_\eta^2}u_{\eta\eta}$$

Letting

$$\begin{aligned}
b_1 &= \frac{x_\xi a_1^*}{(2jk_0 - a_1)} \\
b_2 &= \frac{x_\xi}{y_\eta}\left[\left(a_2 - \frac{y_{\eta\eta}}{y_\eta^2}\right)\frac{1}{(2jk_0 - a_1)} + y_\xi\right] \\
b_3 &= \frac{x_\xi}{(2jk_0 - a_1)y_\eta^2}
\end{aligned}$$

we express the PDE as

$$\boxed{u_\xi = b_1 u + b_2 u_\eta + b_3 u_{\eta\eta}} \tag{43}$$

The normal derivative on a $\eta =$ constant line is

$$\begin{aligned}
u_\nu(\eta = \text{const.}) &= \sqrt{\frac{g_{11}}{g}}u_\eta - \frac{g_{12}}{\sqrt{g g_{11}}}u_\xi \\
&= \frac{\sqrt{x_\xi^2 + y_\xi^2}}{x_\xi y_\eta - y_\xi x_\eta}u_\eta - \frac{x_\xi x_\eta + y_\xi y_\eta}{\sqrt{x_\xi^2 + y_\xi^2}(x_\xi y_\eta - y_\xi x_\eta)}u_\xi
\end{aligned} \tag{44}$$

Variation of an arbitrary vector $\vec{r}$ along the η = const. coordinate is

$$\vec{r}_\xi = x_\xi \hat{x} + y_\xi \hat{y} = \vec{a}_\xi$$

The unit vector along the tangent on η = const. is then

$$\hat{s} = \frac{\vec{a}_\xi}{|\vec{a}_\xi|} = \frac{x_\xi \hat{x}}{\sqrt{x_\xi^2 + y_\xi^2}} + \frac{y_\xi \hat{y}}{\sqrt{x_\xi^2 + y_\xi^2}}$$

Defining

$$\begin{aligned} \cos\theta &= \frac{x_\xi}{\sqrt{x_\xi^2 + y_\xi^2}} \\ \sin\theta &= \frac{y_\xi}{\sqrt{x_\xi^2 + y_\xi^2}} \end{aligned}$$

we express (44) as

$$\therefore \quad u_\nu = \frac{u_\eta}{y_\eta \cos\theta - x_\eta \sin\theta} - \frac{(x_\eta \cos\theta + y_\eta \sin\theta) u_\xi}{\sqrt{x_\xi^2 + y_\xi^2}(y_\eta \cos\theta - x_\eta \sin\theta)}$$

With these substitutions, the boundary condition at the bottom boundary $u_\nu + c_1 u = 0$ gets transformed to

$$u_\eta - \frac{(x_\eta \cos\theta + y_\eta \sin\theta)}{\sqrt{x_\xi^2 + y_\xi^2}} u_\xi + (y_\eta \cos\theta - x_\eta \sin\theta) c_1 u = 0 \quad \text{on } \eta = 0 \tag{45}$$

For $x_\eta = 0$ and using $\sqrt{x_\xi^2 + y_\xi^2} = x_\eta / \cos\theta$, we get

$$\boxed{u_\eta - \frac{y_\eta}{x_\xi} \sin\theta \cos\theta u_\xi + c_1 y_\eta \cos\theta u = 0 \quad @ \quad \eta = 0} \tag{46}$$

Finally at the top boundary, we have

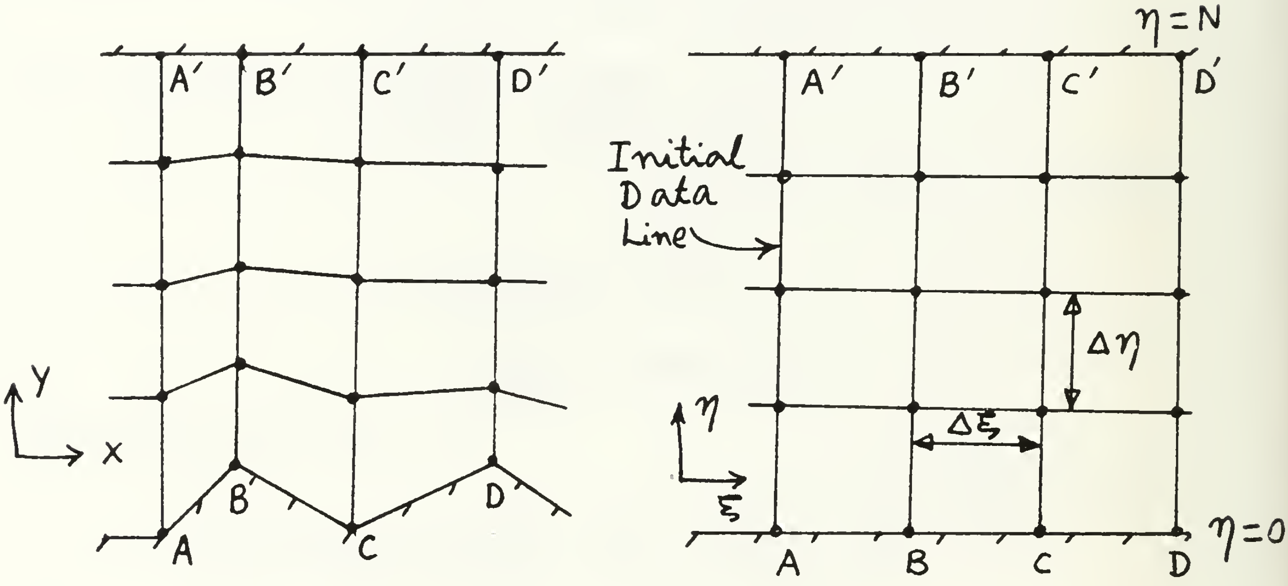
$$u_\eta / y_\eta + r u = s$$

or

$$\boxed{u_\eta(\xi_m, N) + r(\xi_m, N) y_\eta(\xi_m) u(\xi_m, N) = y_\eta s(\xi_m, N)} \tag{47}$$

8. Generation of Curvilinear Coordinate System

Consider a piece-wise linear ground profile and a horizontal upper boundary

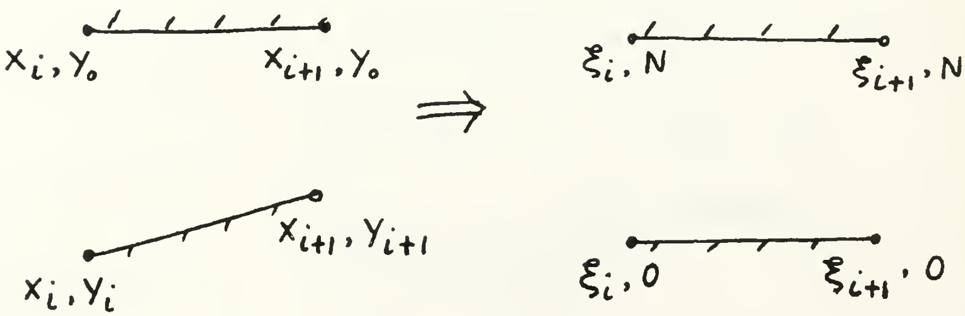


Non-Rectangular,
Uniform Mesh

Physical Domain

Rectangular,
Uniform Mesh

Computational Domain



We generate the (x, y) coordinates of an interior point by using a linear interpolation.

Letting $\Delta x_i = (x_{i+1} - x_i)$, $\Delta y_i = (y_{i+1} - y_i)$, and $\Delta \xi_i = (\xi_{i+1} - \xi_i)$, we write

$$\left.\begin{aligned} x &= x_i + \frac{\Delta x_i}{\Delta \xi_i}(\xi - \xi_i) \\ y &= \left(1 - \frac{\eta}{N}\right)\left[y_i + \frac{\Delta y_i}{\Delta \xi_i}(\xi - \xi_i)\right] + \frac{\eta}{N} y_0 \end{aligned}\right\} \begin{array}{l} \xi_i \le \xi \le \xi_{i+1} \\ 0 \le \eta \le N \end{array} \tag{48}$$

At any interior point, $\xi_i < \xi < \xi_{i+1}$, the various metrics are evaluated as

$$\begin{aligned} x_\xi &= \frac{\Delta x_i}{\Delta \xi_i}, \qquad x_\eta = 0 \\ y_\xi &= \left(1 - \frac{\eta}{N}\right)\frac{\Delta y_i}{\Delta \xi_i}, \qquad y_\eta = \frac{1}{N}\left[y_0 - y_i - \frac{\Delta y_i}{\Delta \xi_i}(\xi - \xi_i)\right], \quad y_{\eta\eta} = 0 \end{aligned} \tag{49}$$

At the boundary points, we use the central difference formulas to arrive at

$$x_\xi(\xi = \xi_i) = \frac{x_{i+2} - x_i}{\xi_{i+2} - \xi_i} = \frac{\Delta x_{i+1} + \Delta x_i}{\Delta \xi_{i+1} + \Delta \xi_i} \tag{50}$$

$$y_\xi(\xi = \xi_i) = \left(1 - \frac{\eta}{N}\right)\frac{\Delta y_{i+1} + \Delta y_i}{\Delta \xi_{i+1} + \Delta \xi_i} \tag{51}$$

Note that the analytical expressions yield a discontinuous value for these derivatives at the boundary points. We use the analytical expressions only to generate grid points and use the central difference formulas to arrive at the derivatives w.r.t. ξ. In other words, once the grid points are generated on the lines AA', BB', ..., etc., we assume that the space is smoothly connected through the grid points. In the numerical implementation using Crank-Nicolson implicit scheme [6], the metrics are needed at the midpoint w.r.t. ξ, i.e., at $\xi = \xi_i + \Delta\xi_i/2$ and the interior point formulas are applicable. For a uniform mesh in the computational domain, $\Delta \xi_i = 1$, $\eta = q$, $q = 0, 1, 2, \cdots, N$

$$x_\xi = \Delta x_i \tag{52}$$

$$y_\xi = \left(1 - \frac{q}{N}\right)\Delta y_i \implies y_\xi(q+1) - y_\xi(q) = -\frac{\Delta y_i}{N} \tag{53}$$

$$y_\eta = \frac{1}{N}\left(y_0 - \frac{y_{i+1} + y_i}{2}\right) \tag{54}$$

$$\begin{aligned} y &= \left(1 - \frac{q}{N}\right)\left(\frac{y_{i+1} + y_i}{2}\right) + \frac{q}{N} y_0 \\ &= \frac{y_{i+1} + y_i}{2} + q y_\eta = y_0 + (q - N) y_\eta \end{aligned} \tag{55}$$

so that $y(q+1) - y(q) = y_\eta$.

9. Numerical Implementation by Crank-Nicolson Scheme

Consider the narrow angle PE together with the boundary condition

$$u_\xi = b_1 u + b_2 u_\eta + b_3 u_{\eta\eta} \qquad \text{(PDE)}$$

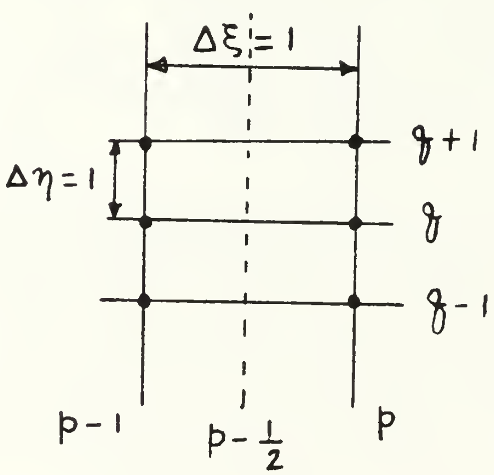
$$u_\eta - \left(\frac{y_\eta}{x_\xi}\sin\theta\cos\theta\right) u_\xi + c_1 y_\eta \cos\theta u = 0 \quad \text{at } \eta = 0 \qquad (\mathrm{BC}_2)$$

$$u_\eta + r y_\eta u = y_\eta\ s \quad \text{at } \eta = N \qquad (\mathrm{BC}_1)$$

with

$$u_\eta(0, N) = 0$$

We would like to implement the above using a Crank-Nicolson implicit scheme [6]



We use the notation $u_q^p = u(\xi_p, \eta_q) = u(x_p, y_q)$

The various derivatives assuming $\Delta\xi = \Delta\eta = 1$ are

$$\begin{aligned}
u_\xi(\xi_{p-1/2}, \eta_q) &= u_q^p - u_q^{p-1} \\
u_\eta\left(p - \frac{1}{2}, q\right) &= u_\eta(\xi_{p-1/2}, \eta_q) \approx \frac{1}{2}\left[u_\eta(\xi_{p-1}, \eta_q) + u_\eta(\xi_p, \eta_q)\right] \\
&= \frac{1}{4}\left[u_{q+1}^{p-1} - u_{q-1}^{p-1} + u_{q+1}^p - u_{q-1}^p\right]
\end{aligned}$$

or

$$u_\eta\left(p - \frac{1}{2}, q\right) = \frac{1}{4}\left[u_{q+1}^{p-1} + u_{q+1}^p - \left(u_{q-1}^{p-1} + u_{q-1}^p\right)\right]$$

$$u_{\eta\eta}\left(p-\frac{1}{2},q\right)=\frac{1}{2}\left[u_{q+1}^{p}-2u_{q}^{p}+u_{q-1}^{p}+u_{q+1}^{p-1}-2u_{q}^{p-1}+u_{q-1}^{p-1}\right]$$

Substituting into the PDE, we get

$$\begin{aligned}
u_q^p-u_q^{p-1} &= b_1\left(p-\frac{1}{2},q\right)\frac{u_q^p+u_q^{p-1}}{2}\\
&\quad+\frac{b_2\left(p-\frac{1}{2},q\right)}{4}\left(u_{q+1}^{p-1}+u_{q+1}^{p}-u_{q-1}^{p}-u_{q-1}^{p-1}\right)\\
&\quad\frac{b_3\left(p-\frac{1}{2},q\right)}{2}\left(u_{q+1}^{p}-2u_{q}^{p}+u_{q-1}^{p}+u_{q+1}^{p-1}-2u_{q}^{p-1}+u_{q-1}^{p-1}\right)
\end{aligned}$$

Rearranging the terms we get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{b_2}{2}+b_3\right)u_{q+1}^{p}-\left(1+b_3-\frac{b_1}{2}\right)u_q^p+\frac{1}{2}\left(b_3-\frac{b_2}{2}\right)u_{q-1}^{p}\\
&+\frac{1}{2}\left(\frac{b_2}{2}+b_3\right)u_{q+1}^{p-1}+\left(1-b_3+\frac{b_1}{2}\right)u_q^{p-1}+\frac{1}{2}\left(b_3-\frac{b_2}{2}\right)u_{q-1}^{p-1}=0
\end{aligned}$$

Now let

$$\begin{aligned}
\alpha &= \frac{1}{2}\left(b_3+\frac{1}{2}b_2\right)_q^{p-1/2}\\
\beta &= \left(b_3-\frac{b_1}{2}\right)_q^{p-1/2}\\
\gamma &= \frac{1}{2}\left(b_3-\frac{b_2}{2}\right)_q^{p-1/2}
\end{aligned}$$

We then have for $p=1,2,\cdots$, $q=1,2,\cdots,N-1$

$$\alpha u_{q+1}^{p}-(1+\beta)u_q^p+\gamma u_{q-1}^{p}=-\alpha u_{q+1}^{p-1}-(1-\beta)u_q^{p-1}-\gamma u_{q-1}^{p-1} \tag{56}$$

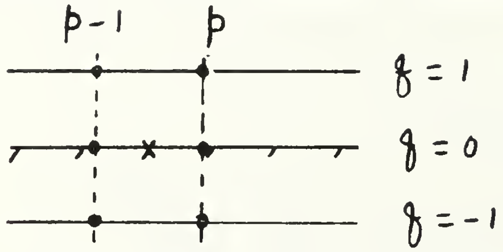
However, we will extend the applicability of this equation over $q=0,1,\cdots N$ to accomodate the derivative boundary conditions on the lower and upper boundaries. For $q=0$, we have

$$\alpha u_1^p-(1+\beta)u_0^p+\gamma u_{-1}^{p}=-\alpha u_1^{p-1}-(1-\beta)u_0^{p-1}-\gamma u_{-1}^{p-1} \tag{57}$$

for $q=N$, we have

$$\alpha u_{N+1}^{p}-(1+\beta)u_N^p+\gamma u_{N-1}^{p}=-\alpha u_{N+1}^{p-1}-(1-\beta)u_N^{p-1}-\gamma u_{N-1}^{p-1} \tag{58}$$

From the BC_2 at the lower boundary we have



$$\frac{1}{2}\left[\frac{u_1^{p-1}-u_{-1}^{p-1}}{2}+\frac{u_1^p-u_{-1}^p}{2}\right]-\left(\frac{y_\eta}{x_\xi}\sin\theta\cos\theta\right)^{p-1/2}\left(u_0^p-u_0^{p-1}\right)$$

$$+\left(c_1 y_\eta\cos\theta\right)_0^{p-1/2}\left(\frac{u_0^p-u_0^{p-1}}{2}\right)=0$$

Multiplying this with $4(\gamma)_0^{p-1/2}$ and adding to (57)

$$(\alpha+\gamma)u_1^p-\left(1+\beta-2\gamma\left[c_1 y_\eta\cos\theta-2\frac{y_\eta}{x_\xi}\sin\theta\cos\theta\right]\right)u_0^p$$

$$=-(\alpha+\gamma)u_1^{p-1}-\left(1-\beta+2\gamma\left[c_1 y_\eta\cos\theta+2\frac{y_\eta}{x_\xi}\sin\theta\cos\theta\right]\right)u_0^{p-1}$$

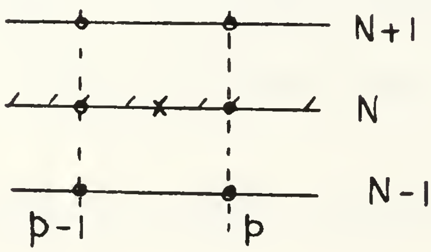
Letting

$$\begin{aligned}
\alpha' &= \alpha+\gamma \\
\beta' &= \beta-2\gamma y_\eta\cos\theta\left(c_1-\frac{2\sin\theta}{x_\xi}\right) \\
\beta'' &= \beta-2\gamma y_\eta\cos\theta\left(c_1+\frac{2\sin\theta}{x_\xi}\right)
\end{aligned}$$

we may write the above equation as

$$\alpha' u_1^p-(1+\beta')u_0^p=-\alpha' u_1^{p-1}-(1-\beta'')u_0^{p-1} \tag{59}$$

From the BC_1 at the top boundary we have



$$\frac{1}{2}\left[\frac{u_{N+1}^{p-1}-u_{N-1}^{p-1}}{2}+\frac{u_{N+1}^{p}-u_{N-1}^{p}}{2}\right]+\frac{r^p+r^{p-1}}{2}y_\eta\frac{u_N^p+u_N^{p-1}}{2}=y_\eta\frac{s^p+s^{p-1}}{2}$$

Multiplying this with $-4(\alpha)_N^{p-1/2}$ and adding to (58), we get

$$-\left(1+\beta+[r^p+r^{p-1}]y_\eta\alpha\right)u_N^p+(\gamma+4\alpha)u_{N-1}^p$$

$$=-\left(1-\beta-[r^p+r^{p-1}]y_\eta\alpha\right)u_N^{p-1}-(\gamma+4\alpha)u_{N-1}^{p-1}-2y_\eta(s^p+s^{p-1})$$

Letting

$$\begin{aligned}\lambda &= \beta+y_\eta\alpha[r^p+r^{p-1}]\\ \gamma' &= \gamma+4\alpha\end{aligned}$$

we write the above equation as

$$(1+\lambda)u_N^p-\gamma' u_{N-1}^p=(1-\lambda)u_N^{p-1}+\gamma' u_{N-1}^{p-1}+2y_\eta(r^p+r^{p-1}) \tag{60}$$

We augment the equations given in (56) for $q=1,2,\ldots,N-1$ with (59) and (60) for $q=0$ and N, respectively to define for $q=0,1,2,\ldots,N$. The system of equations so defined can be expressed as a matrix equation of the form

$$\begin{bmatrix} X & X & & & & \\ X & X & X & & & \\ & X & X & X & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & & & X & X \end{bmatrix}\begin{bmatrix} u_0^p \\ u_1^p \\ \vdots \\ \vdots \\ \vdots \\ u_N^p \end{bmatrix}=\begin{bmatrix} X & X & & & & \\ X & X & X & & & \\ & X & X & X & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & & & X & X \end{bmatrix}\begin{bmatrix} u_0^{p-1} \\ u_1^{p-1} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{p-1} \end{bmatrix} \tag{61}$$

where X denotes a non-zero entry. The tridiagonal matrix on the left hand side of (61) can be inverted efficiently to yield a solution on line $\xi=\xi_p$ in terms of the field values on the line $\xi=\xi_{p-1}$. Equation (61) can be used to march forward in range starting from initial data specified on $\xi=0$.

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